MATH2050C Assignment 13

No need to hand in any problem.

Supplementary Problems

1. Show that for $a > 0$, $\lim_{x\to\infty} \dfrac{x^a}{e^x} = 0$ and $\lim_{x\to-\infty} |x|^a e^x = 0$.
2. Show that for $a > 0$, $\lim_{x\to\infty} \dfrac{\ln x}{x^a} = 0$.
3. Determine the domain(s) of definition and continuity of the function given by the formula $\sqrt{\dfrac{x^2-4}{x+3}}$.
4. Determine the domain(s) of definition and continuity of the function given by the formula $\mathrm{sgn}(x^2-1)$.
5. Determine the domain(s) of definition and continuity of the function given by the formula $\ln(-x^2+2x+3)$.

See next page

The Exponential Function and Powers

From the discussion in Notes 12, we list the properties of the exponential function below.

Theorem 13.1 The exponential function E is a strictly increasing, continuous function from $(-\infty, \infty)$ onto $(0, \infty)$. Moreover, for $n \geq 1$,

$$\lim_{x\to\infty} \frac{x^n}{E(x)} = 0 \ , \quad \lim_{x\to-\infty} |x|^n E(x) = 0 \ .$$

It remains to prove its growth behavior at infinity. Indeed, for $x > 0$, from

$$E(x) = 1 + x + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots > \frac{x^{n+1}}{(n+1)!},$$

we get $0 < x^n/E(x) < (n+1)!x^{-1} \to 0$ as $x \to \infty$. For $x < 0$, using $E(x)E(-x) = 1$, we have $|x|^n E(x) = (-x)^n/E(-x) \to 0$ as $-x \to \infty$, done.

Since E is strictly increasing and continuous, its inverse function, called the logarithmic function $\ln x$, is also strictly increasing and continuous on $(0, \infty)$.From the properties of the exponential function, we deduce

Theorem 13.2 The logarithmic function $\ln x$ is strictly increasing and continuous from $(0, \infty)$ to $(-\infty, \infty)$. It satisfies

1. $E(\ln x) = x, \ x \in (0, \infty); \quad \ln(E(x)) = x, \ x \in (-\infty, \infty)$.
2. $\ln xy = \ln x + \ln y$.
3. For $n \geq 1$, $\lim_{x\to\infty} \ln x/x^n = 0$

After introducing powers of numbers below, the number n in (3) can be replaced by any positive a.

We use the logarithmic function to define the power of x. For $x > 0$ and $a \in \mathbb{R}$, define

$$x^a = E(a \ln x).$$

Immediately we get

$$\ln x^a = a \ln x \ .$$

Letting $e = E(1) = 2.718\cdots$, we also get $e^x = E(x \ln e) = E(x \ln E(1)) = E(x)$. From now on we could use e^x to replace $E(x)$.

Theorem 13.3 For $x, y > 0$ and $a, b \in \mathbb{R}$,

1. $(x^a)^b = x^{ab} = (x^b)^a$.
2. $x^a x^b = x^{a+b}$.
3. $(xy)^a = x^a y^a$.

In particular, this theorem implies rules like $(e^a)^b = e^{ab}$ and $e^a e^b = e^{a+b}$.

Proof (1) $(x^a)^b \equiv (E(a\ln x))^b \equiv E(b\ln E(a\ln x)) = E(ab\ln x) = x^{ab}$.
(2) $(x^a)(x^b) \equiv E(a\ln x)E(b\ln x) = E(a\ln x + b\ln x) = E((a+b)\ln x) \equiv x^{a+b}$.
(3) $(xy)^a \equiv E(a\ln xy) = E(a\ln x + a\ln y) = E(a\ln x)E(a\ln y) = x^a y^a$.

Since x^a is previously defined when a is a rational number, we need to show that the new definition is consistent with the old one. To establish this we first note

Theorem 13.4 For $m, n \in \mathbb{Z}$ and $x > 0$,

1. $(x^m)^n = x^{mn} = (x^n)^m$.
2. $x^m x^n = x^{m+n}$.
3. $(xy)^n = x^n y^n$.

Here for a negative exponent $n < 0$, x^n means $(x^{-1})^{-n}$ where x^{-1} is the multiplicative inverse of x. The elementary proof of this theorem is left to the reader.

For $x > 0$ and $a = m/n$ we define $x^a = (x^{1/n})^m$ (the old definition). We have

$$x^{m/n} \equiv (x^{1/n})^m = (x^m)^{1/n}. \tag{1}$$

Indeed, by Theorem 13.4 (1),

$$((x^{1/n})^m)^n = (x^{1/n})^{mn} = (x^{1/n})^{nm} = ((x^{1/n})^n)^m = x^m ,$$

which shows the n-th power of $(x^{1/n})^m$ is x^m, so (1) holds.

Now, let $a = m/n$, the n-th power of $E(\frac{m}{n}\ln x)$ is

$$E(\frac{m}{n}\ln x)^n = E(\frac{m}{n}\ln x)\cdots E(\frac{m}{n}\ln x) = E(n\frac{m}{n}\ln x) = E(m\ln x) = E(\ln x)^m = x^m,$$

so $E(\frac{m}{n}\ln x) = (x^m)^{1/n} = x^{m/n}$ where (1) has been used in the last step.

Elementary Functions

So far we have encountered functions like polynomials, rational functions, powers, the exponential function, the logarithmic function, and sine/cosine functions. Any function obtained from several steps of algebraic operations, composition of functions or taking inverse is called an elementary functions. Here are some examples:

$$\sqrt{1+\ln x}, \quad \frac{e^{-x^2}(x^2-3)}{1-\sin x}, \quad \mathrm{sgn}\Big(\frac{1-x}{2-x}\ln x\Big), \quad (\tan(x^6+3x-e^x))^{2/3}, \cdots .$$

The domain of definition and the domain of continuity of an elementary function must be determined case by case. Here we examine two examples.

Example 13.1 The function given by the formula $\sqrt{\frac{x+1}{x-5}}$. The square root function is defined and continuous on $[0, \infty)$. We find those x so that $(x+1)/(x-5)$ belongs to $[0, \infty)$. After some consideration, the set $\{x : (x+1)/(x-5) \in [0, \infty)\}$ is given by $E = (-\infty, -1] \bigcup (5, \infty)\}$. We conclude that the domain of definition and continuity of this function is the set E.

Example 13.2 The function $\sqrt{1 + \ln x}$. $1 + \ln x \geq 0$ iff $\ln x \geq -1$ or $x \geq e^{-1}$. Therefore, the domain of definition and continuity of this function is $\{x : x \geq e^{-1}\}$.